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The electromagnetic field of elementary time-dependent toroidal sources

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Abstract. The radiation field of toroidal-like time-dependent current configurations is investigated. The infinitesimal time-dependent configurations are found outside which the electromagnetic strengths disappear but the potentials survive. For a number of time dependences, their finite radiationless counterparts can be found. In these cases topologically non-trivial (unremovable by a gauge transformation) electromagnetic potentials exist outside sources. The well-defined rule obtained for constructing time-dependent infinitesimal sources suggests the existence of finite non-trivial radiationless sources with an arbitrary time dependence. The latter can be used to carry out time-dependent Aharonov–Bohm-like experiments and to transfer the information. Using the Neumann–Helmholtz parametrization of the current density we represent the time-dependent electromagnetic field in a form convenient for applications.

1. Introduction

Interest in time-dependent currents flowing in toroidal coils arose from the following remark made by James Clerk Maxwell in his memoir *On physical lines of force* [1]:

'Let B, figure 3, be a circular ring of uniform section, lapped uniformly with covered wire. It may be shewn that if an electric current is passed through this wire, a magnet placed within the coil of wire will be strongly affected, but no magnetic effect will be produced on any external point. The effect will be that of magnet bent round till its two poles are in contact.

If the coil is properly made, no effect on a magnet placed outside it can be discovered, whether the current is kept constant or made to vary in strength; but if a conducting wire C be made to embrace the ring any number of times, an electromotive force will act on this wire whenever the current in the coil is made to vary; and if the circuit be closed, there will be an actual current in the wire C.'

The figure 3 mentioned in this passage shows the torus with a poloidal winding on its surface. At the present time, it is known that, in general, this Maxwell assertion is not correct. It turns out that for a time-dependent current in a toroidal coil the electromagnetic field strengths appear outside it. Qualitatively this was shown by Mitkevich [2] and Page [3]. The corresponding experiments were performed by Mitkevich [2], Ryazanov [4], Bartlett and Ward [5] and many others. Quantitative results were obtained in [6] where

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the electromagnetic fields were evaluated for a number of time dependences for current flowing in a toroidal coil. After all, experimentalists widely use toroidal transformers for their own purposes without philosophizing on the subject. The sole exception for which Maxwell's claim holds is the current which increases linearly in time which flows in the toroidal coil. In this case $\boldsymbol{H} = 0$ and $\boldsymbol{E}$ is independent of time outside the torus (see, e.g., Miller [7]). The question of the energy transfer into the wire C embracing the torus was considered by Heald [8] (the difficulty is that the Poynting vector equals zero for a linearly increasing current).

In a previous paper [9], we studied the electromagnetic field of static toroidal-like configurations, their interactions with an external electromagnetic field and possible physical applications. It is the goal of the present consideration to study non-static current configurations. It would probably be appropriate to explain the meaning of the words 'elementary toroidal sources' in the title of this paper. The words 'toroidal source' mean the poloidal current flowing in the winding of the toroidal solenoid (TS), which in turn may be an element of a more complex configuration. When the dimensions of this configuration tend to zero, we obtain an 'elementary toroidal source'. The reason for the treatment of an elementary toroidal source is that it leads to a considerable simplification of the theoretical consideration. A TS with finite dimensions has a number of non-trivial topological properties (see, e.g., reviews [10]). Suppose that these properties survive when the dimensions of the TS tend to zero. Thus, if we find some interesting property of the elementary toroidal source, there is a chance that it might survive in the finite toroidal configuration. This is confirmed for the simplest toroidal configurations for which analytical solutions can be found. As an example, we mention the configuration consisting of a TS with a linearly increasing current flowing in its winding and the double charged layer lying at the hole of the TS [9]. Outside this configuration, the electromagnetic strengths disappear but a non-trivial (i.e. unremovable by gauge transformation) time-dependent vector potential (VP) survives. Thus, the possibility of performing a time-dependent Aharonov–Bohm-like experiment arises. However, the linear time dependence of the current is unrealistic. It is the aim of this paper to find elementary charge-current configurations possessing the radiationless properties mentioned above but with an arbitrary time dependence.

The plan of our exposition is as follows. The radiation of elementary time-dependent toroidal-like configurations, in the winding of which a time-dependent current flows, is studied in section 2. It turns out that two different branches of these configurations generate essentially different electromagnetic fields. However, the current sources in the same branch generate the same electromagnetic field if their time dependences are properly adjusted. In section 3 we give an example of an elementary radiationless charge-current source having the property that electromagnetic field strengths disappear outside it, but that the time-dependent potentials survive there. Extended toroidal-like current sources are considered in section 4. By using the Neumann–Helmholtz parametrization for the current density convenient formulae for the time-dependent electromagnetic fields are obtained. Using these, more general elementary radiationless charge-current sources of different multipolarities are constructed in section 5. These elementary configurations have finite counterparts. Those which can be treated analytically are radiationless and have non-trivial electromagnetic potentials outside them. Although the electromagnetic field of more complicated finite configurations cannot be obtained in a closed form, the electromagnetic field of their infinitesimal analogues can. The well prescribed rule for the construction of these elementary radiationless configurations found in section 5 suggests that their finite radiationless counterparts will also possess non-trivial electromagnetic potentials. A short discussion of the results obtained and a summary are given in sections 6 and 7, respectively.

2. Radiation of elementary toroidal sources

2.1. A pedagogical example: time-dependent circular current

According to the Ampere hypothesis the distribution of the magnetic dipoles $\boldsymbol{M}(\boldsymbol{r})$ is equivalent to the current distribution $J(r) = \operatorname{curl} \boldsymbol{M}(\boldsymbol{r})$. For example, the circular current flowing in the $Z = 0$ plane

$$\boldsymbol{J} = I\boldsymbol{n}_\varphi\delta(\rho - d)\delta(z) \tag{2.1}$$

is equivalent to the magnetized sheet

$$\boldsymbol{M} = I \cdot \boldsymbol{n}_z\Theta(d - \rho)\delta(z) \tag{2.2}$$

lying in the same plane ($\Theta(x)$ is a step function). When the radius d of the circumference along which the current flows tends to zero, the current $\boldsymbol{J}$ becomes ill defined (it is not clear what the vector $\boldsymbol{n}_\varphi$ means at the origin). On the other hand, the vector $\boldsymbol{M}$ is still well defined. In this limit the elementary current (2.1) turns out to be equivalent to the magnetic dipole oriented normally to the plane of this current. It is convenient to introduce $I/\pi d^2$ instead of I in equations (2.1) and (2.2). Then, in the limit $d \to 0$ one gets

$$\boldsymbol{M} = I\boldsymbol{n}\delta^3(\boldsymbol{r}) \qquad (\delta^3(\boldsymbol{r}) = \delta(\rho)\delta(z)/2\pi\rho) \tag{2.3}$$

and

$$\boldsymbol{J} = I \operatorname{curl}(\boldsymbol{n}\delta^3(\boldsymbol{r})). \tag{2.4}$$

Equations (2.3) and (2.4) define the magnetization and current density corresponding to the elementary magnetic dipole. These questions were considered in detail in [9]. Now let the intensity of the elementary current change with time:

$$J_0 = f_0(t) \operatorname{curl}(\boldsymbol{n}\delta^3(\boldsymbol{r})) \tag{2.5}$$

(the factor I is absorbed into f_0). The vector potential corresponding to this current is elementarily obtained:

$$\boldsymbol{A}_0 = -\frac{1}{c^2r^2}D_0(\boldsymbol{r} \times \boldsymbol{n}) \qquad D_k = D(f_k) = \dot{f}_k + \frac{c}{z}f_k. \tag{2.6}$$

From now on the time derivative will be denoted either by a point above the letter or (especially for higher derivatives) by superscripts. For example, $f^{(3)} = \dddot{f} = \mathrm{d}^3 f/\mathrm{d}t^3$. The argument of the f functions, if not indicated, means $t - (r/c)$ everywhere in this section. The electromagnetic field strengths are

$$\boldsymbol{E}_0 = \frac{1}{c^3r^2}(\boldsymbol{r} \times \boldsymbol{n})\dot{D}_0 \qquad \boldsymbol{H}_0 = \frac{(\boldsymbol{r}\boldsymbol{n})}{c^3r^3}\boldsymbol{r}F_0 - \frac{1}{c^3r}\boldsymbol{n}G_0 \tag{2.7}$$

where, for brevity, we put

$$F_k = F(f_k) = f_k^{(2)} + 3\frac{c}{r}\dot{f}_k + 3\frac{c^2}{r^2}f_k$$

$$G_k = G(f_k) = f_k^{(2)} + \frac{c}{r}\dot{f}_k + \frac{c^2}{r^2}f_k.$$

The flux of the electromagnetic energy through the sphere of the radius r is

$$S = \int P_r r^2\,\mathrm{d}\Omega = \frac{2}{3c^5}\dot{D}_0G_0 \qquad \boldsymbol{P} = \frac{c}{4\pi}(\boldsymbol{E}_0 \times \boldsymbol{H}_0). \tag{2.8}$$

This flux is positive for large distances and determined by the second derivative of f_0($S_0 \approx \frac{2}{3c^5}\ddot{f}_0^2$). However, for small distances it may be negative. These results are well known and may be found in many textbooks (see, e.g., Stratton [11]).

2.2. The elementary radiating toroidal solenoid

The case which is next in complexity is the radiation of a current flowing in the winding of an elementary (i.e. infinitely) small toroidal solenoid. According to [9], this elementary current is given by

$$j_1 = f_1(t)\,\mathrm{curl}^{(2)}(n\delta^3(r)) \tag{2.9}$$

where $\mathrm{curl}^2 = \mathrm{curlcurl}$ and n means the normal to the equatorial plane of the TS. The VP and field strengths are equal to

$$\begin{aligned} &\phi_1 = 0 \qquad A_1 = -n\frac{1}{c^3 r}G_1 + \frac{1}{c^3 r^3}r(rn)F_1 \\ &E_1 = n\frac{1}{c^4 r}\dot{G}_1 - \frac{1}{c^4 r^3}r(rn)\dot{F}_1 \qquad H_1 = \frac{1}{c^4 r^2}(r\times n)\ddot{D}_1. \end{aligned} \tag{2.10}$$

In this and the following equations in this section we omit the δ function terms giving the field values at the origin (to which the current is confined). Thus, equations (2.10) are valid everywhere except for the origin.

2.3. More complicated elementary toroidal sources

We consider a hierarchy of TS in which each turn is again a TS. The simplest example is the usual TS (which is obtained by installing an infinitely thin TS in a single turn with the current (2.5) in it). We denote this TS by T_1 (the initial current source (2.5) will be denoted by T_0). The case which is next in complexity is obtained when each turn of T_1 is replaced by an infinitely thin TS with an alternating current in its winding. This current configuration is denoted by T_2. When its dimensions tend to zero, we get [9]

$$j_2 = f_2(t)\,\mathrm{curl}^{(3)}\, n\delta^3(r). \tag{2.11}$$

The corresponding VP and field strengths are given by

$$\begin{aligned} &A_2 = \frac{1}{r^2 c^4}D_2^{(2)}(r\times n) \qquad E_2 = -\frac{1}{r^2 c^5}D_2^{(3)}(r\times n) \\ &H_2 = \frac{1}{c^5 r}nG_2^{(2)} - \frac{1}{c^5 r^3}r(nr)F_2^{(2)}. \end{aligned} \tag{2.12}$$

By comparing equations (2.6), (2.7) with (2.12) we conclude that the electromagnetic fields coincide for the current configurations T_0 and T_2 (everywhere except for the origin) if the following relation between the time-dependent intensities is fulfilled $f_2^{(2)} = -f_0/c^2$. This means, in particular, that the electromagnetic field of the static magnetic dipole (f_0 = constant) coincides with that of the current configuration T_2 if the current in it quadratically varies with time ($f_2 = -f_0 c^2 t^2/2$). It follows from this that the magnetic field of the usual magnetic dipole can be compensated everywhere (except for the origin) by the time-dependent current flowing in T_2. Consider now the periodical currents $f_2 = f_{20}\cos\omega t$ and $f_0 = f_{00}\cos\omega t$. Clearly, the electromagnetic fields of T_0 and T_2 coincide if $f_{20} = f_{00}c^2/\omega^2$. Now we are able to write the electromagnetic field for the point-like toroidal configuration of the arbitrary order. Let

$$j_m = f_m(t)\,\mathrm{curl}^{(m+1)}(n\delta^3(r)).$$

Then, for m even ($m = 2k$, $k \geqslant 0$)

$$\begin{aligned} A_{2k} &= \frac{(-1)^{k+1}}{c^{2k+2}}\frac{1}{r^2}(r \times n)D_{2k}^{(2k)} \\ E_{2k} &= \frac{(-1)^k}{c^{2k+3}}\frac{1}{r^2}(r \times n)D_{2k}^{(2k+1)} \\ H_{2k} &= \frac{(-1)^k}{c^{2k+3}}\left[r\frac{(nr)}{r^3}F_{2k}^{(2k)} - n\frac{1}{r}G_{2k}^{(2k)}\right]. \end{aligned} \tag{2.13}$$

The flux of the electromagnetic energy through the sphere of the radius r is equal to

$$S = \frac{2}{3}\frac{1}{c^{4k+5}}G_{2k}^{(2k)}D_{2k}^{(2k+1)}.$$

On the other hand, for m odd ($m = 2k+1$, $k \geqslant 0$),

$$\begin{aligned} A_{2k+1} &= \frac{(-1)^k}{c^{2k+3}}\left[\frac{1}{r^3}r(rn)F_{2k+1}^{(2k)} - \frac{1}{r}nG_{2k+1}^{(2k)}\right] \\ E_{2k+1} &= \frac{(-1)^{k+1}}{c^{2k+4}}\left[\frac{1}{r^3}r(rn)F_{2k+1}^{(2k+1)} - \frac{1}{r}nG_{2k+1}^{(2k+1)}\right] \\ H_{2k+1} &= \frac{(-1)^k}{c^{2k+4}}\frac{1}{r^2}(r \times n)D_{2k+1}^{(2k+2)} \\ S &= \frac{2}{3}\frac{1}{c^{4k+7}}G_{2k+1}^{(2k+1)}D_{2k+1}^{(2k+2)}. \end{aligned} \tag{2.14}$$

We see that there are two branches of toroidal point-like currents generating essentially different electromagnetic fields. A representative of the first branch is the usual magnetic dipole. The electromagnetic field of the kth member of this family reduces to that of the circular current if the time dependences of these currents are properly adjusted:

$$f_{2k}^{(2k)} = (-1)^k f_0(t)/c^{2k} \qquad (k \geqslant 0). \tag{2.15}$$

We remember that the lower index of the f functions selects a particular member of the first branch, while the upper one means the time derivative. The representative of the second branch is the elementary TS. Again, the electromagnetic fields of this family are the same if the time dependences of currents are properly adjusted:

$$f_{2k+1}^{(2k)} = (-1)^k f_1(t)/c^{2k} \qquad (k \geqslant 0). \tag{2.16}$$

From the equations defining the energy flux it follows that, for high frequencies, the toroidal emitters of higher order are more effective (as the time derivatives of higher orders contribute to the energy flux).

So far we have used the usual TS as a cornerstone for constructing more complicated current configurations. Under the term 'usual' we mean the torus $(\rho - d)^2 + z^2 = R^2$ with the poloidal current flowing on its surface. The VP corresponding to this current falls as r^{-3} at large distances

$$A \sim \frac{3r(rn) - nr^2}{r^5} \qquad \text{for } r \to \infty.$$

Here $\boldsymbol{n}$ is the unit vector normal to the equatorial plane of the TS. It has been shown in [12] that it is possible to distribute the currents inside the torus in such a way as to cancel the leading term ($\sim r^{-3}$) in the expansion of the VP. Then the first non-vanishing term in the expansion of the VP has the form

$$A_i \sim \sum n_j n_k n_l Q^{(4)}_{ijkl}(x)/r^9 \tag{2.17}$$

where $Q^{(4)}_{ijkl}$ has the following symmetric traceless form:

$$Q^{(4)}_{ijkl}(x) = x_i x_j x_k x_l - \tfrac{1}{7}(\delta_{ij}x_k x_l + \delta_{ik}x_j x_l + \delta_{il}x_j x_k + \delta_{jk}x_i x_l + \delta_{jl}x_i x_k + \delta_{kl}x_i x_j)r^2$$
$$+ \tfrac{1}{35}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})r^4.$$

This VP falls like r^{-5} for $r \to \infty$. With this TS taken as a cornerstone and using the procedure described above we can construct a new hierarchy of TSs. This game may be continued further. More complicated current configurations may be found inside the torus for which the VP falls like r^{-7}. This current configuration may, in turn, be used as a cornerstone for the construction of TS installed in one another. These cornerstone current configurations correspond to higher-order toroidal multipoles [12]. At large distances all of them may be symbolically written in the form

$$A^{(l)}_i = \sum Q^{(l)}_{i,i_2,\dots,i_l}(x) n_{i_2} n_{i_3} \dots n_{i_l}/r^{2l+1} \tag{2.18}$$

where $Q^{(l)}_{i_1 \dots i_l}$ is the symmetric traceless form of order l. Clearly, $A^{(l)}_i$ fall as r^{-l-1} for $r \to \infty$. Only the even values of l correspond to the finite configurations of poloidal currents found in [12]. As asymptotic form (2.18) satisfies conditions $\operatorname{div} \boldsymbol{A} = 0$, $\operatorname{curl} \boldsymbol{A} = 0$ for any l, the question arises as to the possible existence of finite current toroidal-like configurations (i.e. ones outside of which $\boldsymbol{E} = \boldsymbol{H} = 0$) corresponding to odd l. So far we have not identified them.

3. On the radiationless sources of electromagnetic fields

Consider the electric dipole oriented in the n direction. Its charge density is

$$\rho_{\rm d} = e[\delta^3(\boldsymbol{r} + a\boldsymbol{n}) - \delta^3(\boldsymbol{r} - a\boldsymbol{n})].$$

For small separation a this reduces to

$$\rho_{\rm d} = 2ea(\boldsymbol{n}\nabla)\delta^3(\boldsymbol{r}).$$

Let the intensity of this dipole change with time:

$$\rho_{\rm d} = f_{\rm d}(t)(\boldsymbol{n}\nabla)\delta^3(\boldsymbol{r})$$

(the factor $2ea$ is absorbed into $f_{\rm d}$). The corresponding current density is given by

$$j_{\rm d} = -\dot{f}_{\rm d}\boldsymbol{n}\delta^3(\boldsymbol{r}).$$

These densities generate the following potentials and field strengths (see, e.g., Weinstein [13]):

$$\phi_{\rm d} = -\frac{1}{cr^2}(\boldsymbol{n}\boldsymbol{r})D_{\rm d} \qquad \boldsymbol{A}_{\rm d} = -\boldsymbol{n}\dot{f}_{\rm d}/rc$$
$$\boldsymbol{H}_{\rm d} = \frac{1}{c^2r^2}(\boldsymbol{r}\times\boldsymbol{n})\dot{D}_{\rm d} \tag{3.1}$$
$$\boldsymbol{E}_{\rm d} = \frac{1}{c^4r}\boldsymbol{n}G_{\rm d} - \frac{1}{c^2r^3}(\boldsymbol{r}\boldsymbol{n})\boldsymbol{r}F_{\rm d}.$$

From a comparison of equations (2.10) and (3.1) we conclude that the field strengths of the time-dependent current flowing in the winding of the infinitely small TS can be compensated by that of the electric dipole if the time dependences are properly adjusted: $f_{\rm d} = -\dot{f}_1/c^2$. Then, the total charge-current densities are:

$$\rho = -\frac{1}{c^2}\dot{f}_1(\boldsymbol{n}\nabla)\delta^3(\boldsymbol{r}) \qquad \boldsymbol{j} = f_1(t)\,\mathrm{curl}^{(2)}\,\boldsymbol{n}\delta^3(\boldsymbol{r}) + \frac{1}{c^2}\ddot{f}_1\boldsymbol{n}\delta^3(\boldsymbol{r}). \tag{3.2}$$

In the surrounding space $\boldsymbol{E} = \boldsymbol{H} = 0$ but the potentials differ from zero:

$$\phi = \frac{1}{c^3r^2}(\boldsymbol{n}\boldsymbol{r})\dot{D}_1$$
$$\boldsymbol{A} = -\frac{1}{c^2r^2}\boldsymbol{n}D_1 + \frac{1}{c^3r^3}\boldsymbol{r}(\boldsymbol{r}\boldsymbol{n})F_1. \tag{3.3}$$

Thus, outside this composite object (electric dipole and TS placed at the same point) there are non-vanishing time-dependent electric and vector potentials despite the disappearance of the field strengths. The simplest example corresponds to f_1 = constant. Then,

$$\phi = 0 \qquad \boldsymbol{A} = f_1[3\boldsymbol{r}(\boldsymbol{n}\boldsymbol{r}) - \boldsymbol{n}r^2]/cr^5 \tag{3.4}$$

which coincides with the VP of the elementary (i.e. infinitely small) static TS. The next-in-complexity case is the composite object consisting of the static electric dipole ($f_{\rm d} = f$ = constant) and the current which linearly change with time in the winding of the TS:

$$\rho = f(\boldsymbol{n}\nabla)\delta^3(\boldsymbol{r}) \qquad \boldsymbol{j} = -c^2 ft\,\mathrm{curl}^{(2)}\,\boldsymbol{n}\delta^3(\boldsymbol{r})$$
$$\boldsymbol{E} = \boldsymbol{H} = 0 \qquad \phi = -f(\boldsymbol{n}\boldsymbol{r})/r^3 \tag{3.5}$$
$$\boldsymbol{A} = -ctf[3\boldsymbol{r}(\boldsymbol{n}\boldsymbol{r}) - r^2\boldsymbol{n}]/r^5.$$

A counterpart of (3.5) with finite dimensions is the current linearly rising with time flowing in the winding of TS and the double charged layer filling the hole of the same TS (see the appendix). Outside this configuration the electromagnetic strengths vanish, but the non-trivial (that is, unremovable by a gauge transformation) VP exists.

Another interesting case is the compensation of the electromagnetic field generated by the oscillating electric dipole by that of the periodical current flowing in the winding of the TS:

$$\rho = \rho_{\rm d}\cos(\boldsymbol{n}\nabla)\delta^3(\boldsymbol{r})f$$
$$\boldsymbol{j} = \boldsymbol{j}_{\rm d} + \boldsymbol{j}_1 = f\omega\sin\omega t\left[\boldsymbol{n}\delta^3(\boldsymbol{r}) - \frac{c^2}{\omega^2}\,\mathrm{curl}^{(2)}\,\boldsymbol{n}\delta^3(\boldsymbol{r})\right]$$
$$\boldsymbol{E} = \boldsymbol{H} = 0 \qquad \phi = \frac{1}{cr^2}(\boldsymbol{r}\boldsymbol{n})f\left[\omega\sin\Omega - \frac{c}{r}\cos\Omega\right] \qquad \Omega = \omega\left(t - \frac{r}{c}\right) \tag{3.6}$$
$$\boldsymbol{A} = \frac{1}{r^2}f\boldsymbol{n}\left(\cos\Omega + \frac{c}{\omega r}\sin\Omega\right) + \frac{\omega}{cr^3}(\boldsymbol{r}\boldsymbol{n})f\boldsymbol{r}\left(\sin\Omega - 3\frac{c}{\omega r}\cos\Omega - 3\frac{c^2}{\omega^2r^2}\sin\Omega\right).$$

It turns out that the field strengths are compensated if the charge density of the electric dipole oscillates in counter-phase with the TS current.

There are several references [14–21] in which the non-radiating sources are treated. In some of them the electromagnetic potentials are zero and, thus, of no interest to us. The time-dependent charge-current densities treated here meet the general non-radiation conditions obtained in the cited references, up until now it was not known whether non-trivial non-radiating time-dependent sources could exist in principle. As far as we know, the first such example has been presented in [9].

Non-trivial time-dependent electromagnetic potentials can be used as a new channel for information transfer (by modulating the phase of the charged-partial wavefunction) and for the performance of time-dependent Aharonov–Bohm-like experiments.

4. The finite toroidal-like configurations

4.1. The Neumann–Helmholtz parametrization for electromagnetic potentials and strengths

Consider now the time-dependent current distribution confined to a finite region of space:

$$\boldsymbol{j}(\boldsymbol{r},t) = f(t)\boldsymbol{j}(\boldsymbol{r}). \tag{4.1}$$

An arbitrary vector function and, in particular, the current distribution can be presented in the form (Neumann–Helmholtz parametrization)

$$\boldsymbol{j}(\boldsymbol{r}) = \nabla\Psi_1 + \mathrm{curl}(\boldsymbol{r}\Psi_2) + \mathrm{curl}^{(2)}(\boldsymbol{r}\Psi_3). \tag{4.2}$$

The VP corresponding to the current density (4.1) is given by

$$\boldsymbol{A} = \nabla a_1 + \mathrm{curl}(\boldsymbol{r}a_2) + \mathrm{curl}^{(2)}(\boldsymbol{r}a_3). \tag{4.3}$$

Clearly, equation (4.3) is the Neumann–Helmholtz parametrization for the VP. The functions entering into it are

$$a_k = \frac{1}{c}I_k \qquad I_k = \int \frac{1}{R} f\left(t - \frac{R}{c}\right)\Psi_k(\boldsymbol{r}')\,\mathrm{d}V'. \tag{4.4}$$

Here $R = |\boldsymbol{r} - \boldsymbol{r}'|$. To be complete, we write the corresponding scalar electric potential

$$\phi = -\frac{1}{c}\dot{I}_1 + 4\pi F(t)\Psi_1(\boldsymbol{r}) + \phi_{\mathrm{stat}} \qquad \left(F(t) = \int^t f(t)\,\mathrm{d}t\right). \tag{4.5}$$

Here the point above I_k means the time derivative, and ϕ_{stat} is the scalar potential originating from the time-independent part of the charge density (if it exists): $\phi_{\mathrm{stat}} = \int \frac{1}{R}\rho_{\mathrm{stat}}(\boldsymbol{r}')\,\mathrm{d}V'$. It is convenient to represent the field strengths in the same form as $\boldsymbol{j}$ and $\boldsymbol{A}$:

$$\begin{aligned} \boldsymbol{E} &= \mathrm{grad}\, e_1 + \mathrm{curl}(\boldsymbol{r}e_2) + \mathrm{curl}^{(2)}(\boldsymbol{r}e_3) \\ \boldsymbol{H} &= \mathrm{grad}\, h_1 + \mathrm{curl}(\boldsymbol{r}h_2) + \mathrm{curl}^{(2)}(\boldsymbol{r}h_3). \end{aligned} \tag{4.6}$$

It turns out that

$$\begin{aligned} &e_1 = -\phi_{\mathrm{stat}} - 4\pi F(t)\Psi_1(\boldsymbol{r}) \qquad e_2 = -\frac{1}{c^2}\dot{I}_2 \qquad e_3 = -\frac{1}{c^2}\dot{I}_3 \\ &h_1 = 0 \qquad h_2 = -\frac{1}{c^3}\ddot{I}_3 + \frac{4\pi}{c}f(t)\Psi_3(\boldsymbol{r}) \qquad h_3 = \frac{1}{c}I_2. \end{aligned} \tag{4.7}$$

These representations are convenient because the potentials and strengths are obtained from relatively simple integrals, their time and space derivatives.

It is known [9] that the functions Ψ_2 and Ψ_3 carry information on the magnetic and toroidal (electric) moments, respectively. Thus, putting $\Psi_2(\boldsymbol{r}) = \Psi_2(r)Y_{lm}(\theta,\varphi)$ and $\Psi_3(\boldsymbol{r}) = \Psi_3(r)Y_{lm}(\theta,\varphi)$ we obtain the formulae describing the radiation of particular magnetic and toroidal (electric) multipoles. The functions Ψ_2 and Ψ_3 define the radial distribution of the current sources. Developing the function $g = f(t - R/c)/R$ over the spherical harmonics

$$g = 4\pi \sum \frac{1}{2l+1} g_l(r, r', t) Y_{lm}(\theta,\varphi) Y^*_{lm}(\theta',\varphi') \tag{4.8}$$

we obtain for the particular lm multipole

$$I_{lm}(\Psi_k) = \frac{4\pi}{c}\frac{1}{2l+1} Y_{lm}(\theta,\varphi) \int g_l(r, r', t)\Psi_k(r')r'^2\,\mathrm{d}r' \tag{4.9}$$

(no sum over l, m here).

4.2. Transition to the point-like limit

Equation (4.9) define the integrals for the finite spatial current distribution. To obtain the point current limit we follow the method used by Rowe [22] for the evaluation of the integral I_1 entering into the definition of ϕ (see equation (4.5)). One simply puts

$$\Psi_k(\boldsymbol{r}) \sim Y_{lm}(-\nabla)\delta^3(\boldsymbol{r}). \tag{4.10}$$

It should be clarified what $Y_{lm}(-\nabla)$ means in the right-hand side of this equation. We write

$$Y_{lm}(x) = r^l Y_{lm}(\theta,\varphi) \tag{4.11}$$

where $Y_{lm}(\theta,\varphi)$ is the usual spherical harmonic. Clearly, $Y_{lm}(x)$ is a homogeneous function (of the order l) in Cartesian variables x, y, z. For example,

$$Y_{20} \sim 2z^2 - x^2 - y^2. \tag{4.12}$$

To obtain $Y_{lm}(-\nabla)$ we change x_i by $(-\partial/\partial x_i)$ in equation (4.11). In particular,

$$Y_{20}(-\nabla) \sim 2\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}. \tag{4.13}$$

Many of the properties of the functions $Y_{lm}(x)$ and their physical applications are collected in [23]. Now we substitute (4.13) into (4.4) and integrate by parts:

$$I_k \sim Y_{lm}(\nabla) f\left(t - \frac{r}{c}\right) \Big/ r. \tag{4.14}$$

Inserting this expression into equations (4.4)–(4.7) we obtain the electromagnetic potentials and strengths describing the elementary source.

5. More general radiationless sources

Having obtained explicit expressions for the extended and point-like sources, we now try to construct the radiationless sources of higher multipolarities. Consider charge and current densities corresponding to the oscillating quadrupole moment:

$$\begin{aligned} \rho_q &= f_q(t)[(\boldsymbol{n}\nabla)^2 - \tfrac{1}{3}\Delta]\delta^3(\boldsymbol{r}) \\ \boldsymbol{j}_q &= -\dot{f}_q(t)[\boldsymbol{n}(\boldsymbol{n}\nabla) - \tfrac{1}{3}\nabla]\delta^3(\boldsymbol{r}). \end{aligned} \tag{5.1}$$

On the other hand, consider current density (4.2) with

$$\begin{aligned} &\Psi_1 = \Psi_2 = 0 \qquad \Psi_3 = [(\boldsymbol{n}\nabla)^2 - \tfrac{1}{3}\Delta]\delta^3(\boldsymbol{r}) \\ &\boldsymbol{j}_c = f_c(t)\,\mathrm{curl}^{(2)}(\boldsymbol{r}\Psi_3). \end{aligned} \tag{5.2}$$

It turns out that the oscillating quadrupole charge-current configuration (5.1) and a pure current configuration (5.2) placed at the same point generate total field strengths equal to zero everywhere except for the origin if the following relation is fulfilled: $f_q = 2\dot{f}_c/c^2$. The total charge-current densities are equal to

$$\begin{aligned} \rho &= \frac{2}{c^2}\dot{f}_c(t)[(\boldsymbol{h}\nabla)^2 - \tfrac{1}{3}\Delta]\delta^3(\boldsymbol{r}) \\ \boldsymbol{j} &= f_c(t)\,\mathrm{curl}^{(2)}(\boldsymbol{r}\Psi_3) - \frac{2}{c^2}\ddot{f}_c[\boldsymbol{n}(\boldsymbol{n}\nabla) - \tfrac{1}{3}\nabla]\delta^3(\boldsymbol{r}). \end{aligned} \tag{5.3}$$

Nevertheless, the electromagnetic potentials are not zero:

$$\begin{aligned} \phi &= \phi_q = \frac{2}{c^4 r^3}[(\boldsymbol{n}\boldsymbol{r})^2 - \tfrac{1}{3}r^2]\dot{F}_c \\ \boldsymbol{A} &= \boldsymbol{A}_q + \boldsymbol{A}_c = -\frac{4}{c^3 r^3}[(\boldsymbol{n}\boldsymbol{r})\boldsymbol{n} - \tfrac{1}{3}\boldsymbol{r}]F_c + \frac{2}{c^4 r^4}\boldsymbol{r}[(\boldsymbol{r}\boldsymbol{n})^2 - \tfrac{1}{3}r^2] \\ &\qquad \times \left(f_c^{(3)} + 6\frac{c}{r}f_c^{(2)} + 15\frac{c^2}{r^2}\dot{f}_c + 15\frac{c^3}{r^3}f_c\right) \\ &\left(F_c = f_c^{(2)} + 3\frac{c}{r}\dot{f}_c + 3\frac{c^2}{r^2}f_c\right). \end{aligned} \tag{5.4}$$

For $f_c = \text{constant}$, $f_q = 0$ we get the following static configuration:

$$\begin{aligned} \boldsymbol{j} &= f_c(t)\,\mathrm{curl}^{(2)}(\boldsymbol{r}\Psi_3) \\ \boldsymbol{A} &= -\frac{12}{cr^5}f_c[(\boldsymbol{n}\boldsymbol{r})\boldsymbol{n} - \tfrac{1}{3}\boldsymbol{r}] + \frac{30}{cr^7}f_c\boldsymbol{r}[(\boldsymbol{n}\boldsymbol{r})^2 - \tfrac{1}{3}r^2]. \end{aligned} \tag{5.5}$$

This VP falling at large distances as r^{-4} corresponds to $l = 3$ in equation (2.18). As we have mentioned, we did not succeed in identifying the finite static current configuration whose infinitesimal limit coincides with (5.5).

The next-in-complexity case corresponds to octupole oscillations of the charge density:

$$\begin{aligned} \rho &= f_q(t)(\boldsymbol{n}\nabla)[(\boldsymbol{n}\nabla)^2 - \tfrac{3}{5}\Delta]\delta^3(\boldsymbol{r}) \\ \boldsymbol{j} &= -\dot{f}_q(t)\boldsymbol{n}[(\boldsymbol{n}\nabla)^2 - \tfrac{3}{5}\Delta]\delta^3(\boldsymbol{r}). \end{aligned} \tag{5.6}$$

The elementary toroidal current distribution giving the same field strengths corresponds to

$$\Psi_1 = \Psi_2 = 0 \qquad \Psi_3 = f_c(t)(\boldsymbol{n}\nabla)[(\boldsymbol{n}\nabla)^2 - \tfrac{3}{5}\Delta]\delta^3(\boldsymbol{r})$$
$$f_q = -3\dot{f}_c/c^2. \tag{5.7}$$

The finite poloidal current distribution whose infinitesimal limit coincides with equation (5.7) was obtained in [12]. The asymptotic behaviour of the corresponding VP is determined by equation (2.17).

Now we are able to write more general radiationless charge-current configurations. The extension of equations (5.1) and (5.6) to an arbitrary multipolarity l is given by

$$\rho_q = f_q(t)(\boldsymbol{v}\nabla)\delta^3(\boldsymbol{r}) \qquad \boldsymbol{j}_q = -\dot{f}_q(t)\boldsymbol{v}\delta^3(\boldsymbol{r}). \tag{5.8}$$

Here $\nabla_i = \partial/\partial x_i$, $\boldsymbol{v}$ is the vector whose Cartesian components are

$$v_i = \sum_{i_2 \dots i_l} Q^{(l)}_{i,i_2 \dots i_l} \nabla_{i_2 \dots} \nabla_{i_l}.$$

$Q^{(l)}_{i_1 \dots i_l}$ is the symmetric traceless form (see equation (2.18)) of the variables n_x, n_y, n_z defining the direction of the fixed 3-vector (this vector will later be identified with the direction of TS axis). The electromagnetic potentials and field strengths corresponding to these densities are

$$\phi_q = (\boldsymbol{v}\nabla)\frac{f_q}{r} \qquad \boldsymbol{A}_q = -\frac{1}{c}\boldsymbol{v}\frac{\dot{f}_q}{r}$$
$$\boldsymbol{E}_q = -\operatorname{grad}(\boldsymbol{v}\nabla)\frac{f_q}{r} + \frac{1}{c^2}\boldsymbol{v}\frac{\ddot{f}_q}{r} \tag{5.9}$$
$$\boldsymbol{H}_q = -\frac{1}{c}(\nabla \times \boldsymbol{v})\frac{\dot{f}_q}{r}$$

(remember that argument of the f functions, if not indicated, means $t - r/c$).

On the other hand, a pure current configuration generalizing equations (5.2) and (5.7) is given by

$$\rho_c = 0 \qquad \boldsymbol{j}_c = f_c(t)\,\mathrm{curl}^{(2)}(\boldsymbol{r}\Psi_3)$$
$$\Psi_3 = (\boldsymbol{v}\nabla)\delta^3(\boldsymbol{r}). \tag{5.10}$$

The corresponding electromagnetic potentials and field strengths are

$$\phi_c = 0$$
$$\boldsymbol{A}_c = -\frac{l}{c}\operatorname{grad}(\boldsymbol{v}\nabla)\frac{\dot{f}_c}{r} + \frac{l}{c^3}\boldsymbol{v}\frac{\dddot{f}_c}{r} + \frac{4\pi}{c}f_c(t)\boldsymbol{r}(\boldsymbol{v}\nabla)\delta^3(\boldsymbol{r})$$
$$\boldsymbol{E}_c = \frac{l}{c^2}\operatorname{grad}(\boldsymbol{v}\nabla)\frac{\ddot{f}_c}{r} - \frac{l}{c^4}\boldsymbol{v}\frac{f_c^{(3)}}{r} - \frac{4\pi}{c^2}\dot{f}_c(t)\boldsymbol{r}(\boldsymbol{v}\nabla)\delta^3(\boldsymbol{r}) \tag{5.11}$$
$$\boldsymbol{H}_c = \frac{l}{c^3}(\nabla \times \boldsymbol{v})\frac{\ddot{f}_c}{r} - \frac{4\pi l}{c}f_c(t)(\boldsymbol{r}\times\nabla)(\boldsymbol{v}\nabla)\delta^3(\boldsymbol{r}).$$

Now we place charge-current densities (5.8) and (5.10) at the same point. It turns out that if $f_q = (l/c^2)\dot{f}_c$ then the total electromagnetic field strengths are everywhere zero except for the origin:

$$\begin{aligned} \boldsymbol{H} &= -\frac{1}{c}4\pi l f_c(t)(\nabla \times \boldsymbol{v})\delta^3(\boldsymbol{r}) \\ \boldsymbol{E} &= \frac{1}{c^2}4\pi l \dot{f}_c(t)\boldsymbol{v}\delta^3(\boldsymbol{r}). \end{aligned} \tag{5.12}$$

Nevertheless, the electromagnetic potentials differ from zero:

$$\begin{aligned} \phi &= -\frac{1}{c}\dot{\chi} \qquad \boldsymbol{A} = \operatorname{grad}\chi - \frac{1}{c}4\pi l f_c(t)\boldsymbol{v}\cdot\delta^3(\boldsymbol{r}) \\ \chi &= -\frac{1}{c}l(\nabla \boldsymbol{v})\frac{f_c}{r}. \end{aligned} \tag{5.13}$$

Evidently, equations (3.2), (3.3), (5.3), (5.4), (5.6) and (5.7) are the particular cases of equations (5.8)–(5.13).

6. Discussion

In a previous section we have found elementary charge-current configurations with the property that electromagnetic strengths, not potentials disappear outside them. Turning to equation (5.13) we observe that outside the source $\boldsymbol{A} = \operatorname{grad}\chi$ and $\phi = -\dot{\chi}/c$, that is, electromagnetic potentials can be presented there as a 4-gradient of a singular function χ. Does this mean that electromagnetic potentials can be eliminated by a gauge transformation? One cannot comment on the topological non-triviality of electromagnetic potentials without going beyond the framework of the elementary source. This is due to the fact that it is not clear what is the topologically non-trivial point-like source. As an illustration consider the vector potential (3.4) of the usual static elementary toroidal solenoid. It turns out that outside the origin (where the TS is placed) the VP may be presented as a gradient of function $\chi = -f_1(\boldsymbol{nr})/r^3$. On the other hand, outside the finite TS (whose infinitesimal counterpart is elementary source (3.4)) the VP cannot be eliminated by the gauge transformation (despite the fact that $\boldsymbol{E} = \boldsymbol{H} = 0$ there). This leads to numerous experimental consequences and, in particular, to the static magnetic Aharonov–Bohm effect. The experiments in which the electrons were scattered on the impenetrable magnetized ring were performed by Tonomura *et al* [24]. Their theoretical description was given in [25].

Now we turn again to equations (5.12), (5.13). We know [12] how to find finite counterparts of the elementary sources (5.10). For time dependences for which VP can be found in a closed form, the prescriptions (5.8)–(5.12) lead to the topologically non-trivial electromagnetic potentials outside the radiationless sources [9]. However, the uniformity of these prescriptions suggests that non-trivial potentials should exist for an arbitrary time dependence. In quantum field theory, the elementary radiationless sources are known under the title 'anapole moments' (or simply, anapoles) (see, e.g., [26–28]). They were introduced by Zeldovich [29]. To the best of our knowledge the non-trivial radiationless sources considered in the previous section are the first concrete realizations of anapoles. It turns out that multipole expansion of the field strengths does not exist in the space surrounding radiationless sources (these field strengths equal zero there). Since the electromagnetic strengths generating by the oscillating charge densities and the elementary toroidal sources are the same (if their time dependences are properly adjusted), particular terms of the multipole expansions defining these strengths coincide and have the double names known in a physical literature as electric (see, e.g., [30]) or toroidal [26] multipoles. Despite the coincidence of the electromagnetic strengths, the corresponding potentials may be physically different. In those cases the multipole expansion of the field strengths does not describe the whole physical situation (since the same multipole expansion of the field strengths corresponds to physically different electromagnetic potentials).

7. Conclusion

We briefly summarize the main results obtained here:

1. The radiation field of toroidal-like current configurations has been investigated. For a given multipole there are two different representatives which generate essentially different electromagnetic fields.

2. Elementary time-dependent charge-current configurations outside of which the electromagnetic field strengths disappear but the potentials survive were found. In the solvable cases their finite-dimensional counterparts have non-trivial (i.e. unremovable by the gauge transformation) electromagnetic potentials outside them. This can be used to perform time-dependent Aharonov–Bohm-like experiments and information transfer (modulating the phase of the charge particle wavefunction).

3. Using the Neumann–Helmholtz parametrization of the current density we present the electromagnetic field of an arbitrary time-dependent charge-current density in a form convenient for applications. The contributions of different multipoles in it are explicitly separated.

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Appendix

Consider the poloidal current on the torus surface $(\rho - d)^2 + t^2 = R^2$ which increases linearly with time: $j = j_0 t$. To parametrize J it is convenient to introduce the coordinates $\tilde{R}$, ψ, $x = (d + R\cos\psi)\cos\varphi$, $y = (d + R\cos\psi)\sin\varphi$, $t = R\sin\psi$. In these coordinates,

$$j_0 = n_\psi \frac{\delta(\tilde{R} - R)}{d + R\cos\psi} \frac{j_0}{R^2}.$$

Here n_ψ is the unit vector tangential to the torus surface

$$n_\psi = n_z \cos\psi - n_\rho \sin\psi.$$

It lies in the $\varphi =$ constant plane and defines the direction of J. It turns out [6] that for this current only the electric strength E differs from zero outside the torus. For simplicity we consider the infinitely thin torus ($R \ll d$). The following representation for the VP is valid [10]:

$$A_x = \frac{\phi_0 t}{4\pi}\frac{\partial^2\alpha}{\partial x\partial z} \qquad A_y = \frac{\phi_0 t}{4\pi}\frac{\partial^2\alpha}{\partial y\partial z} \qquad A_z = \frac{\phi_0 t}{4\pi}\left(\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2}\right).$$

Here

$$\phi_0 = -\frac{4\pi^2}{d}\frac{j_0}{c} \qquad \alpha = \iint \frac{\mathrm{d}x'\,\mathrm{d}y'}{|\boldsymbol{r} - \boldsymbol{r}'|}.$$

The integration in α is performed over the circle $z = 0$, $\rho \leqslant d$ coinciding with the hole of the infinitely thin torus. It was shown in [10] that the VP has singularities nowhere except for the line $z = 0$, $\rho = d$ into which torus T degenerates itself. Outside this line the electromagnetic strengths are

$$\begin{aligned} &\boldsymbol{H} = 0 \qquad F_x = -\frac{1}{4\pi c}\phi_0\frac{\partial^2\alpha}{\partial x\partial z} \qquad E_y = -\frac{1}{4\pi c}\phi_0\frac{\partial^2\alpha}{\partial y\partial z} \\ &E_z = \frac{1}{4\pi c}\phi_0\left(\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2}\right). \end{aligned} \tag{A.1}$$

On the other hand, the electric field produced by two oppositely charged layers $\rho \leqslant d$, $z = \pm\varepsilon$ filling the torus hole is given by

$$\begin{aligned} &E_x = \frac{2e\varepsilon}{\pi d^2}\frac{\partial^2\alpha}{\partial x\partial z} \qquad E_y = \frac{2e\varepsilon}{\pi d^2}\frac{\partial^2\alpha}{\partial y\partial z} \\ &E_z = \frac{2e\varepsilon}{\pi d^2}\frac{\partial^2\alpha}{\partial z^2} = \frac{2e\varepsilon}{\pi d^2}\left(\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2}\right) - \frac{8e\varepsilon}{d^2}\delta(z)\Theta(d-\rho). \end{aligned} \tag{A.2}$$

We see that E_z has a singularity on the circle $z = 0$, $\rho \leqslant d$. From a comparison of equations (A.1) and (A.2) it follows that if $\phi_0 = 8ce\varepsilon/d^2$ then the electric field of the linearly increasing poloidal current is compensated by that of the double layer everywhere except for the position of the layer itself. In this case the electromagnetic potentials and field strengths are equal to

$$\begin{aligned} &\phi = -\frac{1}{4\pi c}\phi_0\frac{\partial\alpha}{\partial z} \qquad A_x = \frac{1}{4\pi}\phi_0 t\frac{\partial^2\alpha}{\partial x\partial z} \qquad A_y = \frac{1}{4\pi}\phi_0 t\frac{\partial^2\alpha}{\partial y\partial z} \\ &A_z = -\frac{\phi_0 t}{4\pi}\left(\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2}\right) \qquad E_x = E_y = 0 \qquad E_z = -\frac{1}{c}\phi_0\delta(z)\Theta(d-\rho) \\ &H_\rho = H_z = 0 \qquad H_\varphi = \phi_0 t\,\delta(z)\,\delta(\rho - d). \end{aligned} \tag{A.3}$$

The Schrödinger equation corresponding to these potentials is

$$i\hbar\frac{\partial\psi}{\partial z} = \left[-\frac{\hbar^2}{2m}\left(\nabla - \frac{ie}{\hbar e}A\right)^2 + e\phi\right]\psi. \tag{A.4}$$

Consider the scattering of charged particles on such a charge-current configuration (to prevent the particle penetration into the torus interior, it can be made impenetrable). Outside it the magnetic field $H = 0$ everywhere, the electric field is also everywhere zero except for the singularity at the torus hole. The static scalar and VPs that are linearly increasing with time differ from zero everywhere. The integral $\oint A_l \, dl$ taken along the closed path passing through the torus hole also grows linearly with time. The question arises as to what extent the electromagnetic potentials can be removed from the Schrödinger equation (A.4). However, first we remember the situation for the usual static magnetic TS without the double charged layer [24, 25]. In this case $\phi = 0$,

$$A_x = \frac{\phi_0}{4\pi}\frac{\partial^2\alpha}{\partial x \partial t} \qquad A_y = \frac{\phi_0}{4\pi}\frac{\partial^2\alpha}{\partial y \partial z} \qquad A_z = -\frac{\phi_0}{4\pi}\left(\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2}\right)$$

(ϕ_0 is the magnetic flux inside the TS). The following gauge transformation

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} - \operatorname{grad}\chi \qquad \psi \to \psi' = \psi \exp(\mathrm{i}e\chi/\hbar c)$$

$$\chi = \frac{1}{4\pi}\phi_0\frac{\partial\alpha}{\partial z}$$

results in

$$A'_x = A'_y = 0 \qquad A'_z = \phi_0\delta(z)\Theta(d-\rho)$$

$$\mathrm{i}\hbar\frac{\partial\psi'}{\partial z} = -\frac{\hbar^2}{2m}\left[\nabla_x^2 + \nabla_y^2 + \left(\nabla_z - \frac{\mathrm{i}e}{\hbar c}\phi_0\delta(z)\cdot\Theta(d-\rho)\right)^2\right]\psi'. \tag{A.5}$$

The VP cannot be removed from equation (A.5) by the gauge transformation and this leads to a shift in the interference picture on the screen installed behind the TS. The corresponding experiments have been performed by Tonomura [24] and their theoretical description is given in [25]. For the treated time-dependent case the gauge transformation which partially eliminates the electromagnetic potentials is

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} - \operatorname{grad}\chi \qquad \phi \to \phi' = \phi + \frac{1}{c}\chi$$

$$\psi \to \psi' = \psi\exp\left(\frac{\mathrm{i}e\chi}{\hbar c}\right) \qquad \chi = \frac{1}{4\pi}\phi_0 t\frac{\partial\alpha}{\partial z}.$$

After this transformation

$$\phi' = A'_x = A'_y = 0 \qquad A'_z = \phi_0 t\delta(z)\Theta(d-\rho)$$

$$E'_z = E_z = -\frac{1}{c}\phi_0\delta(z)\Theta(d-\rho) \qquad H'_\varphi = H_\varphi = \phi_0 t\delta(z)\delta(\rho-d)$$

$$\mathrm{i}\hbar\frac{\partial\psi'}{\partial z} = -\frac{\hbar^2}{2m}\left[\nabla_x^2 + \nabla_y^2 + \left(\nabla_z - \frac{\mathrm{i}e}{\hbar c}\phi_0 t\delta(z)\Theta(d-\rho)\right)^2\right]\psi'. \tag{A.6}$$

Equations (A.5) and (A.6) have essentially the same form. Likewise the static VP cannot be removed from equation (A.5), the time-dependent VP cannot be removed from equation (A.6). This means that an interference picture that changes with time inevitably arises on the screen installed behind the impenetrable TS. The static electric field E filling the torus hole certainly deflects the incoming charged particles (via the Lorentz force). The charged-particle scattering cross section evaluated according to the laws of classical mechanics does not depend upon time. The time dependence of the interference picture is a pure quantum effect. It is due to the time-dependent magnetic flux enclosed in the impenetrable torus. We observe that the effects of the excluded fields (time-dependent magnetic field confined to the impenetrable torus) are observed against a background of accessible ones (the electric field filling torus hole). This agrees with a standard definition of the Aharonov–Bohm effect as observable effects of enclosed (or inaccessible) fields (see, e.g., [24]). For the cylindrical geometry the magnetic time-dependent AB effect was considered recently in [31].

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